Testing monotonicity of a hazard: asymptotic distribution theory

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Abstract

Two new test statistics are introduced to test the null hypotheses that the sampling distribution has an increasing hazard rate on a specified interval $[0, a]$. These statistics are empirical L_1-type distances between the isotonic estimates, which use the monotonicity constraint, and either the empirical distribution function or the empirical cumulative hazard. They measure the excursions of the empirical estimates with respect to the isotonic estimates, due to local non-monotonicity. Asymptotic normality of the test statistics, if the hazard is strictly increasing on $[0, a]$, is established under mild conditions. This is done by first approximating the global empirical distance by an distance with respect to the underlying distribution function. The resulting integral is treated as sum of increasingly many local integrals to which a CLT can be applied. The behavior of the local integrals is determined by a canonical process: the difference between the stochastic process $x \mapsto W(x) + x^2$ where W is standard two-sided Brownian Motion, and its greatest convex minorant.

1   Introduction

One way of characterizing a distribution of an absolutely continuous random variable X that is particularly useful in reliability theory and survival analysis, is by its hazard rate h_0. Suppose X models the failure time of a certain device. The interpretation of the hazard rate is that for small $\epsilon > 0$, $\epsilon h_0(x)$ reflects the probability of failure of the device in the time interval $(x, x + \epsilon]$ given the device was still unimpaired at time x (assuming h_0 is continuous at x). Put differently, $h_0(x)$ represents the level of instantaneous risk of failure of the device at time x, given it still works at time x. A high value reflects high risk, a low value low risk. Lifetimes of devices that are subject to aging can be described by distributions with increasing hazard rate. Locally decreasing hazard rates can be used to model life times of devices that become more reliable with age during a certain period of time.

It is especially this clear interpretation of these qualitative properties of a hazard rate that makes this function a natural characteristic of a survival distribution. The problem of estimating a hazard rate nonparametrically under qualitative (or shape) restrictions gained attention in the sixties of the previous century (see GROENEBOOM AND JONGBLOED (2011B) and the references therein). Also the problem of testing the null hypothesis of constant hazard (exponentiality) against monotonicity of the hazard was studied intensively, see e.g. PROSCHAN AND PYKE (1967). Only quite recently another testing problem, with a "shape constraint" rather than parametric null hypothesis was studied. See also the discussion in companion paper GROENEBOOM AND JONGBLOED (2011A).

In this paper, we consider the asymptotic distribution theory for two integral-type test statistics for the hypothesis that a hazard rate h_0 is monotone on an interval $[0, a]$, for some known $a > 0$. We restrict ourselves to the increasing case; the case of locally decreasing hazard can be considered analogously.

Based on an i.i.d. sample $X_1, \ldots, X_n$ from the distribution associated with H_0, the most natural nonparametric estimator for H_0 without assuming anything on H_0, is the empirical cumulative hazard function given by

$$\mathbb{H}_n(x) = \left\{ \begin{array}{ll} -\log\left\{1 - \mathbb{F}_n(x)\right\}, & x \in \left[0, X_{(n)}\right), \\ \infty, & x \geq X_{(n)} \end{array} \right.$$

where $\mathbb{F}_n$ denotes the empirical distribution function based on $X_1, X_2, \ldots, X_n$. Under the assumption that H_0 is convex on $[0, a]$, the cumulative hazard can be estimated by the greatest convex minorant $\hat{H}_n$ of the empirical cumulative hazard function $\mathbb{H}_n$ on the interval $[0, a]$. Using these two estimators, the following test statistic emerges:

$$T_n = \int_{[0,a]} \left\{\mathbb{H}_n(x-) - \hat{H}_n(x)\right\} d\mathbb{F}_n(x). \tag{1.1}$$

Note that this is the empirical L_1-distance between the two mentioned estimators for the cumulative hazard function w.r.t. the empirical measure $d\mathbb{F}_n$, and that $T_n \geq 0$ since $\hat{H}_n$ is a minorant of $\mathbb{H}_n$. If H_0 is concave on $[0, a]$, both estimators for H_0 will be close to H_0 and T_n will tend to be small (converge to zero a.s. for $n \to \infty$). On the contrary, if h_0 has a region in $[0, a]$ where it is not increasing, $\mathbb{H}_n$ will capture this "non-convexity" of H_0 and converge to H_0 on this region whereas $\hat{H}_n$ will converge to the convex minorant of H_0 on $[0, a]$. Note that $T_n = 0$ if and only if $\hat{H}_n$ coincides with the linear interpolation of the points $(x_{(i)}, \mathbb{H}_n(x_{(i)}-))$ on the range of the data falling in $[0, a]$. One could say that $T_n = 0$ if $\mathbb{H}_n$ is 'as convex as it can be on $[0, a]$', being an increasing right continuous step function. This is the reason for taking $\mathbb{H}_n(x-)$ instead of $\mathbb{H}_n(x)$ in (1.1). A similar reasoning can be held for another test statistic,

$$U_n = \int_{[0,a)} \left\{\mathbb{F}_n(x-) - \hat{F}_n(x)\right\} d\mathbb{F}_n(x), \text{ where } \hat{F}_n(x) = 1 - \exp(-\hat{H}_n(x)). \tag{1.2}$$

An advantage of this definition is that U_n is less sensitive to possible problems that can occur with large values of $\mathbb{H}_n$.

The main result of this paper concerns the asymptotic distribution of T_n and U_n: under certain assumptions

$$n^{5/6} \left\{T_n - E\tilde{T}_n\right\} \xrightarrow{\mathcal{D}} N\left(0, \sigma_{H_0}^2\right) \text{ and } n^{5/6} \left\{U_n - EU_n\right\} \xrightarrow{\mathcal{D}} N\left(0, \sigma_{F_0}^2\right), \tag{1.3}$$

where $\tilde{T}_n$ is a modified version of T_n, see Theorems 4.1 and 4.2. Here $\sigma_{H_0}^2$ and $\sigma_{F_0}^2$ are constants depending on f_0. Results of a similar flavor were established in, e.g., KULIKOV AND LOPUHAÄ (2008) for the difference between the empirical distribution function and its concave majorant.

The basic idea of the proof is to approximate the integral in the test statistic as sum of increasingly many local integrals, using the crucial localization Lemma 3.4, and to apply a Central Limit Theorem to the components that arise in this way. The behavior of the local integrals is determined by a canonical process, the difference between a Brownian motion with parabolic drift and its convex minorant. Relevant properties of this process are derived in section 2. In section 3, a statistic related to T_n (where the integral is taken w.r.t. F_0 rather than $\mathbb{F}_n$) is closely approximated by an integral involving the independent increments of Brownian motion. Moreover, the

resulting integral is represented as a sum of local integrals using a "big blocks separated by small blocks" construction as introduced in ROSENBLATT (1956). The local integrals over the big blocks reduce to the processes considered in section 2. Finally, because the local integrals are based on the independent increments of a Brownian motion process, a CLT can be applied to obtain the first result in the spirit of (1.3). In section 4, the main results of the paper are established by showing that the differences between the integrals w.r.t. $d\mathbb{F}_n$ and dF_0 are sufficiently small.

2 Asymptotic local problem

Consider the process

$$x \mapsto V(x) = W(x) + x^2, \, x \in \mathbb{R} \tag{2.4}$$

with W standard two-sided Brownian motion on $\mathbb{R}$. Then, for $c > 0$, define the functional Q_c by

$$Q_c = \int_0^c \{V(x) - C(x)\} \, dx, \tag{2.5}$$

where C is the greatest convex minorant of V on $\mathbb{R}$. For a picture of the process V and its greatest convex minorant, restricted to the interval $[-2, 2]$, see Figure 1. We have the following result.

Theorem 2.1
$$c^{-1/2} \{Q_c - cE|C(0)|\} \xrightarrow{\mathcal{D}} N(0, \sigma^2), \, c \to \infty,$$

where $C(0)$ is the value of the greatest convex minorant C of the process V at zero, and

$$\sigma^2 = 2 \int_0^\infty \text{covar}(-C(0), V(x) - C(x)) \, dx.$$

All moments of $c^{-1/2} \{Q_c - cE|C(0)|\}$ exist and (in particular) the fourth moment is uniformly bounded in c and converges to the fourth moment of the normal $N(0, \sigma^2)$ distribution, as $c \to \infty$.

In the proof we will use the following lemma, which is proved in the appendix.

Lemma 2.1 *For the process V defined in (2.4), there exist positive constants c and c' such that for all $u \geq 0$*

$$P\left(\min_{x \notin [-u,u]} V(x) \leq 0\right) \leq ce^{-c'u^3}.$$

Proof of Theorem 2.1. It follows from the results in GROENEBOOM (1989) that the process

$$V(x) - C(x), \, x \in \mathbb{R}, \tag{2.6}$$

is stationary. In fact, the process touches zero at changes of slope of C and behaves between these touches of zero as an excursion of a Brownian motion path above a parabola of the form

$$\phi(x) = s - (x - a)^2, \, x \in \mathbb{R},$$

where ϕ is a parabola touching two local minima of Brownian motion, and where the (random) values a and s depend on the Brownian motion path. Defining

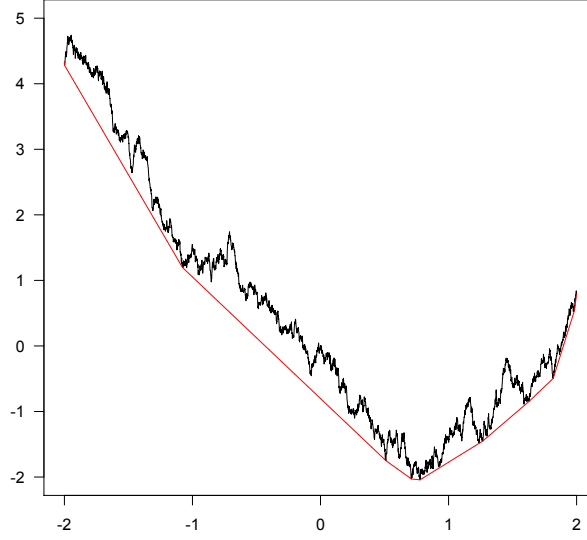$$D_k = \int_k^{k+1} \{V(x) - C(x)\} \, dx, \, k \in \mathbb{Z},$$

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Figure 1: The greatest convex minorant of $W(x) + x^2$, restricted to $[-2, 2]$.

we get a stationary sequence of random variables, and the stationarity of the process (2.6) yields:

$$ED_k = \int_k^{k+1} E\left\{V(x) - C(x)\right\} \, dx = E|C(0)|.$$

Moreover, all moments of D_k exist. This follows from the fact that

$$\max_{x \in [0,1]} \left\{V(x) - C(x)\right\}$$

has a distribution with tails which die out faster than exponentially. To see this, note that, $\forall u \geq 0$,

$$\mathbb{P}\left\{\max_{x \in [0,1]} \left\{V(x) - C(x)\right\} \geq M\right\} \leq \mathbb{P}\left\{\max_{x \in [0,1]} V(x) \geq \tfrac{1}{2}M\right\} + \mathbb{P}\left\{\min_{x \in \mathbb{R}} C(x) \leq -\tfrac{1}{2}M\right\}$$

$$\leq \mathbb{P}\left\{\max_{x \in [0,1]} W(x) \geq \tfrac{1}{2}M - 1\right\} + \mathbb{P}\left\{\min_{x \in \mathbb{R}} V(x) \leq -\tfrac{1}{2}M\right\}$$

$$\leq \sqrt{\frac{2}{\pi}} \int_{\frac{1}{2}M-1}^{\infty} e^{-\frac{1}{2}x^2} \, dx + \mathbb{P}\left\{\min_{x \in [-u,u]} W(x) \leq -\tfrac{1}{2}M\right\} + \mathbb{P}\left\{\min_{x \notin [-u,u]} V(x) \leq 0\right\}. \tag{2.7}$$

The first term on the right hand side is bounded by $c\exp\{-c'M^2/4\}$ for some $c, c' > 0$. By Lemma 2.1 we have for the third term:

$$P\left(\min_{x \notin [-u,u]} V(x) \leq 0\right) \leq ce^{-c'u^3}$$

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for constants $c, c' > 0$. For the second term in (2.7), we get by Brownian scaling,

$$\mathbb{P}\left\{\min_{x\in[-u,u]} W(x) \leq -M/2\right\} = \mathbb{P}\left\{\min_{x\in[-1,1]} W(ux) \leq -M/2\right\}$$

$$= \mathbb{P}\left\{\min_{x\in[-1,1]} u^{-1/2}W(ux) \leq -u^{-1/2}M/2\right\} = \mathbb{P}\left\{\max_{x\in[-1,1]} W(x) \geq u^{-1/2}M/2\right\}$$

$$\leq 2\sqrt{\frac{2}{\pi}} \int_{u^{-1/2}M/2}^{\infty} e^{-\frac{1}{2}x^2}\, dx \leq 2\sqrt{\frac{2}{\pi}} \frac{2\sqrt{u}}{M} \exp\left\{-\tfrac{1}{8}M^2/u\right\}.$$

Hence, taking $u = \sqrt{M}$ in the second and third term in (2.7) and observing that the first term is of lower order, we obtain

$$\mathbb{P}\{\max_{x\in[0,1]} \{V(x) - C(x)\} \geq M\} \leq c_1 e^{-c_2 M^{3/2}} \tag{2.8}$$

for constants $c_1, c_2 > 0$.

Now let $\tau(a)$ be defined by:

$$\tau(a) = \operatorname{argmin}_{x\in\mathbb{R}} \left\{W(x) + (x-a)^2\right\}.$$

The (stationary) process $a \mapsto \tau(a) - a$ is studied in GROENEBOOM (1989) and it follows from the results, given there, that there exist positive constants c_1 and c_2 such that

$$|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| \leq c_1 e^{-c_2 m^3},$$

for events A and B such that

$$A \in \sigma\left\{\tau(a) : a \leq 0\right\}, \qquad B \in \sigma\left\{\tau(a) : a \geq m\right\}.$$

This implies that there also exist positive constants c_1 and c_2 such that

$$|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| \leq c_1 e^{-c_2 m^3}, \tag{2.9}$$

for events A and B such that

$$A \in \sigma\left\{V(x) - C(x) : x \leq 0\right\}, \qquad B \in \sigma\left\{V(x) - C(x) : x \geq m\right\}.$$

So we can apply Theorem 18.5.3 in IBRAGIMOW AND LINNIK (1971), p. 347, yielding that

$$c^{-1/2}\left\{Q_c - cE|C(0)|\right\} \xrightarrow{\mathcal{D}} N(0, \sigma^2),$$

where

$$\sigma^2 = \operatorname{var}(D_0) + 2\sum_{k=1}^{\infty} \operatorname{covar}(D_0, D_k).$$

Using the stationarity of the process (2.6) again, we obtain

$$\sigma^2 = 2\int_0^{\infty} \operatorname{covar}(-C(0), V(x) - C(x))\, dx.$$

The last statement of the theorem follows from (2.8) and (2.9). $\qquad\square$

We will also need the following extension of Theorem 2.1.

Theorem 2.2 *Let C_c be the greatest convex minorant on $[0,c]$ of the process*

$$V(x), \ x \in [0,c].$$

Note that C_c is not the restriction of C to $[0,c]$, since C is globally defined on $\mathbb{R}$, and C_c is the greatest convex minorant of the process V on $[0,c]$, and only defined on $[0,c]$.

  (i) Let, for $c > 4$, the interval I_c be defined by

$$I_c = \left[\sqrt{c}, c - \sqrt{c}\right].$$

  Then:

$$c^{-1/2}\left\{\int_{I_c} \{V(x) - C_c(x)\}\, dx - E\int_{I_c} \{V(x) - C_c(x)\}\, dx\right\} \xrightarrow{\mathcal{D}} N(0, \sigma^2), \ c \to \infty, \quad (2.10)$$

  where σ^2 is defined as in Theorem 2.1.

  (ii) Relation (2.10) also holds if the interval I_c is given by:

$$I_c = \left[0, c - \sqrt{c}\right] \ \text{or} \ I_c = \left[\sqrt{c}, c\right].$$

  (iii) For any choice of I_c in (i) or (ii), the fourth moment of

$$c^{-1/2}\left\{\int_{I_c} \{V(x) - C_c(x)\}\, dx - E\int_{I_c} \{V(x) - C_c(x)\}\, dx\right\}$$

  is uniformly bounded in c, and converges to the fourth moment of a normal $N(0, \sigma^2)$ distribution, as $c \to \infty$.

Proof. (i). The probability that C_c is different from C on the interval I_c is less than or equal to

$$k_1 \exp\left\{-k_2 c^{3/2}\right\},$$

for constants $k_1, k_2 > 0$. The proof of this is analogous to the proof of Lemma 3.4 in the next section. Hence, if K_c denotes the event that $C_c \not\equiv C$ on I_c, we get:

$$E\int_{I_c} |C_c(x) - C(x)|\, dx \leq \left\{\int_{I_c} E\{V(x) - C(x)\}^2\, dx\right\}^{1/2} \mathbb{P}(K_c)^{1/2} = O\left(c^{1/2} e^{-kc^{3/2}}\right), \ c \to \infty,$$

for some $k > 0$. Hence:

$$c^{-1/2}\left\{\int_{I_c} \{V(x) - C_c(x)\}\, dx - E\int_{I_c} \{V(x) - C_c(x)\}\, dx\right\}$$
$$= c^{-1/2}\left\{\int_{I_c} \{V(x) - C(x)\}\, dx - E\int_{I_c} \{V(x) - C(x)\}\, dx\right\} + O_p\left(ce^{-kc^{3/2}}\right),$$

and the statement now follows.

(ii). We can repeat the argument on the interval $[0, \sqrt{c}]$, and apply the argument used in (i) on the subinterval $I_c' = [c^{1/4}, \sqrt{c} - c^{1/4}]$ (but leaving C_c as it was defined in (i)). This yields:

$$c^{-1/4}\left\{\int_{I_c'} \{V(x) - C_c(x)\}\, dx - E\int_{I_c'} \{V(x) - C_c(x)\}\, dx\right\} \xrightarrow{\mathcal{D}} N(0, \sigma^2), \ c \to \infty,$$

implying:

$$c^{-1/2}\left\{\int_{I_c'}\{V(x)-C_c(x)\}\ dx - E\int_{I_c'}\{V(x)-C_c(x)\}\ dx\right\} \xrightarrow{p} 0,\ c\to\infty,$$

Moreover,

$$c^{-1/2}\int_{[0,c^{1/4}]} E\,|V(x)-C_c(x)|\ dx = O\left(c^{-1/4}\right),\ c\to\infty.$$

The statement now follows for the first choice of the interval I_c in (ii). For the second choice of I_c the argument is similar.

(iii). Let I_c be as in (i). Then:

$$c^{-2}E\left\{\int_{I_c}\{V(x)-C_c(x)\}\ dx - E\int_{I_c}\{V(x)-C_c(x)\}\ dx\right\}^4$$

$$= c^{-2}E\left\{\int_{I_c}\{V(x)-C(x)\}\ dx - E\int_{I_c}\{V(x)-C(x)\}\ dx\right\}^4 + O\left(e^{-kc^{3/2}}\right),$$

for some $k>0$, and the statement now follows from Theorem 2.1, (2.8) and (2.9) and the fact that

$$\left(\frac{c}{c-2\sqrt{c}}\right)^2 \to 1,\ c\to\infty.$$

If, for example, $I_c = [0, c-\sqrt{c}]$, we write

$$\int_{I_c}\{V(x)-C_c(x)\}\ dx - E\int_{I_c}\{V(x)-C_c(x)\}\ dx = A_c + B_c,$$

where

$$A_c = \int_{[0,\sqrt{c}]}\{V(x)-C_c(x)\}\ dx - E\int_{[0,\sqrt{c}]}\{V(x)-C_c(x)\}\ dx$$

and

$$B_c = \int_{[\sqrt{c},c-\sqrt{c}]}\{V(x)-C_c(x)\}\ dx - E\int_{[\sqrt{c},c-\sqrt{c}]}\{V(x)-C_c(x)\}\ dx.$$

Hence we get:

$$c^{-2}E\left\{\int_{I_c}\{V(x)-C_c(x)\}\ dx - E\int_{I_c}\{V(x)-C_c(x)\}\ dx\right\}^4$$

$$= c^{-2}EB_c^4 + c^{-2}\left\{4EB_c^3A_c + 6EB_c^2A_c^2 + 4EB_cA_c^3 + EA_c^4\right\}.$$

We have:

$$c^{-2}EA_c^4 = \left(\frac{\sqrt{c}}{c}\right)^2 c^{-1}EA_c^4 = O\left(c^{-1}\right),$$

and similarly, using the Cauchy-Schwarz inequality,

$$c^{-2}EB_cA_c^3 = Ec^{-3/2}A_c^3c^{-1/2}B_c \le \sqrt{Ec^{-3}A_c^6}\sqrt{Ec^{-1}B_c^2} = O\left(c^{-3/4}\right).$$

Continuing in this way, we find that the only non-vanishing term is $c^{-2}EB_c^4$. The statement now follows from what we proved for $I_c = [\sqrt{c}, c-\sqrt{c}]$. $\qquad\square$

We finally also need the following extension of Theorem 2.2.

Theorem 2.3 *Let F_c, G_c and H_c be twice differentiable increasing functions on $[0, c]$, with continuous derivatives f_c, g_c and h_c, respectively, satisfying*

$$F_c(x) = f_c(0)x(1 + o(1)), \qquad G_c(x) = g_c(0)x(1 + o(1)), \qquad H_c(x) = \tfrac{1}{2}h_c'(0)x^2(1 + o(1)), \ c \to \infty,$$

where the $o(1)$ term is uniform in x. We assume that $f_c(0)$, $g_c(0)$, $h_c(0)$ and $h_c'(0)$ are positive and stay away from zero and ∞, as $c \to \infty$, where $h_c'(0)$ denotes the right derivative of h_c at zero. Let C_c be the greatest convex minorant on $[0, c]$ of the process

$$V_c(x) = H_c(x) + W(G_c(x)), \ x \in [0, c].$$

Moreover, let S_c be defined by

$$S_c(x) = V_c(x) - C_c(x), \ x \in [0, c].$$

Then:

(i) *Let, for $c > 4$, the interval I_c be defined by*

$$I_c = \left[\sqrt{c}, c - \sqrt{c}\right].$$

Then:

$$c^{-1}E \int_{I_c} S_c(x)\, dF_c(x) \sim \frac{g_c(0)^{2/3}f_c(0)}{\left(\tfrac{1}{2}h_c'(0)\right)^{1/3}}E|C(0)|, \ \mathrm{var}\left(c^{-1/2}\int_{I_c} S_c(x)\, dF_c(x)\right) \sim \sigma_c^2, \ c \to \infty,$$

(2.11)

where

$$\sigma_c^2 = \frac{g_c(0)^{5/3}f_c(0)^2}{\left(\tfrac{1}{2}h_c'(0)\right)^{4/3}}\sigma^2,$$

(2.12)

and C and σ^2 are defined as in Theorem 2.1. Moreover, the fourth moment of

$$c^{-1/2}\int_{I_c} \{S_c(x) - ES_c(x)\}\, dF_c(x)$$

is uniformly bounded, and satisfies:

$$E\left(c^{-1/2}\int_{I_c} \{S_c(x) - ES_c(x)\}\, dF_c(x)\right)^4 \sim M_c^{(4)}, \ c \to \infty,$$

(2.13)

where $M_c^{(4)}$ denotes the fourth moment of a normal $N(0, \sigma_c^2)$ distribution.

(ii) *Relations (2.11) and (2.13) also hold if the interval I_c is given by:*

$$I_c = \left[0, c - \sqrt{c}\right], \ I_c = \left[\sqrt{c}, c\right] \ \text{or} \ I_c = [0, c].$$

Proof. Since the proof proceeds along the lines of the proofs of Theorems 2.1 and 2.2, we only pay attention to the new type of scaling which is present in the process

$$x \mapsto \tfrac{1}{2}h_c'(0)x^2 + W(g_c(0)x), \ x \in [0, c],$$

which replaces the process

$$x \mapsto V(x) = x^2 + W(x), \ x \in [0, c].$$

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Let $a, b > 0$. By Brownian scaling, the process

$$x \mapsto ax^2 + W(bx), \; x \in [0, c], \tag{2.14}$$

has the same distribution as the process

$$x \mapsto a^{-1/3}b^{2/3}\left\{\left(a^{2/3}b^{-1/3}x\right)^2 + W(a^{2/3}b^{-1/3}x)\right\}, \; x \in [0, c]. \tag{2.15}$$

Hence, if $C_{a,b}$ is the greatest convex minorant of the process given in (2.14) and $\tilde{C}_{a,b}$ of the process given in (2.15) we get:

$$\int_0^c \left\{ax^2 + W(bx) - C_{a,b}(x)\right\} f_c(0) \, dx$$

$$\stackrel{\mathcal{D}}{=} a^{-1/3}b^{2/3} f_c(0) \int_0^c \left\{\left(a^{2/3}b^{-1/3}x\right)^2 + W(a^{2/3}b^{-1/3}x) - a^{1/3}b^{-2/3}\tilde{C}_{a,b}(x)\right\} dx$$

$$= \frac{bf_c(0)}{a} \int_0^{a^{2/3}b^{-1/3}c} \left\{u^2 + W(u) - a^{1/3}b^{-2/3}\tilde{C}_{a,b}\left(a^{-2/3}b^{1/3}u\right)\right\} du$$

$$= \frac{bf_c(0)}{a} \int_0^{a^{2/3}b^{-1/3}c} \left\{u^2 + W(u) - C_c(u)\right\} du,$$

where C_c is the greatest convex minorant of the process

$$u \mapsto u^2 + W(u), \; u \in \left[0, a^{2/3}b^{-1/3}c\right].$$

Thus, for $c \to \infty$

$$c^{-1}E \int_0^c \left\{ax^2 + W(bx) - C_{a,b}(x)\right\} f_c(0) \, dx \sim \frac{b^{2/3} f_c(0)}{a^{1/3}} E|C(0)|.$$

Using that $a = \frac{1}{2}h_c'(0)$, $b = g_c(0)$, (2.11) follows. Moreover,

$$\text{var}\left(c^{-1/2} \int_0^c \left\{ax^2 + W(bx) - C_{a,b}(x)\right\} f_c(0) \, du\right)$$

$$= \frac{b^2 f_c(0)^2}{a^2 c} \text{var}\left(\int_0^{a^{2/3}b^{-1/3}c} \left\{u^2 + W(u) - C_c(u)\right\} du\right)$$

$$= \frac{b^{5/3} f_c(0)^2}{a^{4/3}} \text{var}\left(\frac{1}{\sqrt{b^{-1/3}a^{2/3}c}} \int_0^{a^{2/3}b^{-1/3}c} \left\{u^2 + W(u) - C_c(u)\right\} du\right)$$

$$\sim \frac{b^{5/3} f_c(0)^2}{a^{4/3}} \sigma^2 \, c \to \infty.$$

Taking $a = \frac{1}{2}h_c'(0)$, $b = g_c(0)$ now yields (2.12). □

3 Embedding and first central limit result

In this section, a central limit result is established for the quantity

$$n^{5/6} \int_0^a \left\{\mathbb{H}_n(x) - \hat{H}_n(x) - \mu_n\right\} dF_0(x) = n^{5/6} \int_0^a \left\{\mathbb{H}_n(x-) - \hat{H}_n(x) - \mu_n\right\} dF_0(x), \tag{3.16}$$

where μ_n denotes a centering sequence to be specified below. This result is the first step to be taken in order to obtain the limit result for T_n defined in (1.1) (where the integral is taken with respect to $d\mathbb{F}_n$ rather than dF_0). In order to derive the limiting distribution of (3.16), we first replace the process $\mathbb{H}_n(x) - \hat{H}_n(x)$ by

$$x \mapsto H_0(x) + \frac{E_n(x)}{\sqrt{n}\{1 - F_0(x)} - \tilde{H}_n(x), \ x \in [0, a].$$

where E_n is the empirical process $\sqrt{n}\{\mathbb{F}_n - F_0\}$ and $\tilde{H}_n$ is the greatest convex minorant of the process

$$x \mapsto H_0(x) + n^{-1/2}E_n(x)/\{1 - F_0(x)\}, \ x \in [0, a]. \tag{3.17}$$

Next we use the strong approximation of the empirical process by a Brownian bridge B_n, yielding the approximation

$$x \mapsto H_0(x) + \frac{n^{-1/2}B_n(F_0(x))}{1 - F_0(x)}, \ x \in [0, a],$$

the process (3.17). This process is distributed as

$$x \mapsto H_0(x) + n^{-1/2}W\left(\frac{F_0(x)}{1 - F_0(x)}\right), \ x \in [0, a], \tag{3.18}$$

where W is standard Brownian motion on $[0, \infty)$. Next, the interval $[0, a]$ is split up in so-called big blocks separated by small blocks. The local contributions to the integral over the big blocks can be treated using the results of section 2.

The first lemma to be proved states a contraction property for convex minorants that will be used repeatedly in the sequel. It is related to Marshall's Lemma in the theory of isotonic regression.

Lemma 3.1 *Let f and g be bounded functions on an interval $I \subset \mathbb{R}$ and let C_f and C_g be their greatest convex minorants, respectively. Then:*

$$\sup_{x \in I} |C_f(x) - C_g(x)| \leq \sup_{x \in I} |f(x) - g(x)|.$$

Proof. Using that $f \geq g - \sup_{u \in I} |f(u) - g(u)|$ and that $g \geq C_g$ by definition, it follows that $f \geq C_g - \sup_{u \in I} |f(u) - g(u)|$. Since the right hand side is convex, this means that it is *a* convex minorant of f on I. Hence, it lies below the greatest convex minorant C_f of f on I:

$$C_f(x) \geq C_g(x) - \sup_{u \in I} |f(u) - g(u)|, \ x \in I.$$

Since this inequality also holds with f and g interchanged, the result follows. $\qquad\square$

We now consider the functional

$$\int_{[0,a]} \left\{ \mathbb{H}_n(x) - \hat{H}_n(x) \right\} dF_0(x)$$

$$= \int_{[0,a]} \left\{ H_0(x) - \log\left(1 - \frac{E_n(x)}{\sqrt{n}\{1 - F_0(x)\}}\right) - \hat{H}_n(x) \right\} dF_0(x), \tag{3.19}$$

where $E_n = \sqrt{n}\{\mathbb{F}_n - F_0\}$ is the empirical process. The following lemma enables us to dispense with the logarithms.

Lemma 3.2 *Let $\tilde{H}_n$ be the greatest convex minorant of the process*

$$x \mapsto H_0(x) + \frac{E_n(x)}{\sqrt{n}\{1 - F_0(x)\}}, \ x \in [0, a],$$

where $F_0(a) < 1$. Then:

(i)

$$\int_{[0,a]} \left| \mathbb{H}_n(x) - H_0(x) - \frac{E_n(x)}{\sqrt{n}\{1 - F_0(x)\}} \right| dF_0(x) = O_p\left(n^{-1}\right).$$

(ii)

$$\int_{[0,a]} \left| \hat{H}_n(x) - \tilde{H}_n(x) \right| dF_0(x) = O_p\left(n^{-1}\right).$$

Proof. (i). Let A_n denote the event

$$\left| \sup_{x \in [0,a]} \frac{E_n(x)}{\sqrt{n}\{1 - F_0(x)\}} \right| \le \frac{1}{2}.$$

Then, by a well-known result in large deviation theory ("Chernoff's theorem"), we have

$$\mathbb{P}\left(A_n^c\right) = O\left(e^{-nc}\right),$$

for a constant $c > 0$. If A_n occurs, we can expand the logarithm, which yields:

$$-\log\left\{1 - \frac{E_n(x)}{\sqrt{n}\{1 - F_0(x)\}}\right\} = \frac{E_n(x)}{\sqrt{n}\{1 - F_0(x)\}} + n^{-1}O\left(\sup_{x \in [0,a]} |E_n(x)|\right),$$

and (i) now follows.

(ii). This follows from Lemma 3.1 and the argument of the proof of (i). $\qquad\square$

We shall prove below that

$$n^{5/6} \int_{[0,a]} \left\{ H_0(x) + \frac{E_n(x)}{\sqrt{n}\{1 - F_0(x)\}} - \tilde{H}_n(x) - E\left\{ H_0(x) + \frac{E_n(x)}{\sqrt{n}\{1 - F_0(x)\}} - \tilde{H}_n(x) \right\} \right\} dF_0(x)$$

converges in distribution to a normal distribution, which, together with Lemma 3.2, implies that

$$n^{5/6} \int_{[0,a]} \left\{ \mathbb{H}_n(x) - \hat{H}_n(x) - E\left\{ H_0(x) + \frac{E_n(x)}{\sqrt{n}\{1 - F_0(x)\}} - \tilde{H}_n(x) \right\} \right\} dF_0(x)$$

converges to the same normal distribution.

Remark 3.1 We avoid taking the expectation of

$$\mathbb{H}_n(x) - \hat{H}_n(x),$$

since $\mathbb{H}_n$ is infinite with a positive (but vanishing) probability on $[0, a]$, as is $\hat{H}_n$. This happens when the empirical distribution function $\mathbb{F}_n$ reaches the value 1 on $[0, a]$.

By Theorem 3 of KÓMLOS, MAJOR AND TUSNÁDY (1975) we can construct Brownian bridges B_n on the same sample space as $\mathbb{F}_n$ such that

$$Y_n = \sup_{x \in [0,a]} \frac{n^{1/2} |E_n(x) - B_n(F_0(x))|}{2 \vee \log n}$$

is a random variable with with $EY_n \leq C < \infty$ for all n. Hence, for $n \geq 2$:

$$0 \leq E \sup_{x \in [0,a]} n^{-1/2} \left| \frac{E_n(x)}{1 - F_0(x)} - \frac{B_n(F_0(x))}{1 - F_0(x)} \right| \leq \frac{EY_n \log n}{n(1 - F_0(a))} = O\left(\frac{\log n}{n} \right). \tag{3.20}$$

We now have the following result.

Lemma 3.3 *Let $\tilde{E}_n$ be defined by*

$$\tilde{E}_n(x) = \frac{B_n(F_0(x))}{1 - F_0(x)}, \ x \in [0, a], \tag{3.21}$$

and let C_n^B be the greatest convex minorant of

$$H_0(x) + n^{-1/2} \tilde{E}_n(x), \ x \in [0, a].$$

Then

$$\int_{[0,a]} \left\{ \mathbb{H}_n(x) - \hat{H}_n(x) \right\} \, dF_0(x)$$

$$= \int_{[0,a]} \left\{ H_0(x) + n^{-1/2} \tilde{E}_n(x) - C_n^B(x) \right\} \, dF_0(x) + O_p\left(\frac{\log n}{n} \right). \tag{3.22}$$

Proof. The result immediately follows from (3.20) and Lemmas 3.1 and 3.2(i). □

We now note that the process

$$x \mapsto \frac{B(F_0(x))}{1 - F_0(x)}, \ x \in [0, a],$$

has the same distribution as the process

$$x \mapsto V_n(x) \stackrel{\text{def}}{=} H_0(x) + n^{-1/2} W\left(\frac{F_0(x)}{1 - F_0(x)} \right), \ x \in [0, a], \tag{3.23}$$

where W is standard Brownian motion on $\mathbb{R}_+$. So, if C_n is the greatest convex minorant of the process

$$x \mapsto H_0(x) + n^{-1/2} W\left(\frac{F_0(x)}{1 - F_0(x)} \right), \ x \in [0, a],$$

we have:

$$\int_{[0,a]} \left\{ H_0(x) + n^{-1/2} \tilde{E}_n(x) - C_n^B(x) \right\} \, dF_0(x)$$

$$\stackrel{\mathcal{D}}{=} \int_{[0,a]} \left\{ H_0(x) + n^{-1/2} W\left(\frac{F_0(x)}{1 - F_0(x)} \right) - C_n(x) \right\} \, dF_0(x). \tag{3.24}$$

Theorem 3.1 *Let h_0 be strictly positive on $[0, a]$, with a strictly positive continuous derivative h_0' on $(0, a)$, which also has a strictly positive right limit at 0 and a strictly positive left limit at a. Moreover, let S_n be defined by*

$$S_n(x) = H_0(x) + n^{-1/2}W\left(\frac{F_0(x)}{1 - F_0(x)}\right) - C_n(x), \ x \in [0, a], \tag{3.25}$$

where C_n is the greatest convex minorant of V_n defined in (3.23) and let D_n be defined by

$$D_n = \int_0^a S_n(x) \, dF_0(x). \tag{3.26}$$

Finally, let $C(0)$ and σ^2 be defined as in Theorem 2.1. Then:

$$n^{5/6}\left\{D_n - ED_n\right\} \xrightarrow{\mathcal{D}} N(0, \sigma_{H_0}^2), \ n \to \infty,$$

where

$$n^{2/3}ED_n \to E|C(0)| \int_0^a \left(\frac{2h_0(t)f_0(t)}{h_0'(t)}\right)^{1/3} dH_0(t), \tag{3.27}$$

and

$$\sigma_{H_0}^2 = 2^{4/3}\sigma^2 \int_0^a \frac{h_0(t)^2\{h_0(t)f_0(t)\}^{1/3}}{h_0'(t)^{4/3}} \, dH_0(t) \tag{3.28}$$

The following corollary is immediate from the preceding.

Corollary 3.1 *Let h_0 be strictly positive on $[0, a]$, with a strictly positive continuous derivative h_0' on $(0, a)$, which also has a strictly positive right limit at 0 and a strictly positive left limit at a. Then:*

$$n^{5/6}\left\{\int_0^a \left\{\mathbb{H}_n(x) - \hat{H}_n(x)\right\} dF_0(x) - ED_n\right\} \xrightarrow{\mathcal{D}} N(0, \sigma_{H_0}^2), \ n \to \infty,$$

where ED_n and $\sigma_{H_0}^2$ are defined as in Theorem 3.1.

For the proof of Theorem 3.1 we divide the interval $[0, a]$ into m_n intervals $I_{n,k}$ with (equal) length of order $n^{-1/3}\log n$ (big blocks), separated by intervals $J_{n,k}$ ($k = 2, 3, \ldots, m_n$) with length of order $2n^{-1/3}\sqrt{\log n}$ (small blocks). The small interval $J_{n,1}$ to the left of $I_{n,1}$ has half the length of the other separating blocks as has the small interval J_{n,m_n+1} to the right of I_{n,m_n}. Hence,

$$[0, a] = J_{n,1} \cup I_{n,1} \cup J_{n,2} \cup I_{n,2} \cdots \cup J_{n,m_n} \cup I_{n,m_n} \cup J_{n,m_n+1}.$$

For $k = 2, 3, \ldots, m_n$, let $\tilde{J}_{n,k}$ be the interval with the same right endpoint as $J_{n,k}$ with half the length of $J_{n,k}$ and take $\tilde{J}_{n,1} = J_{n,1}$. For $k = 1, 2, \ldots, m_n - 1$ let $\bar{J}_{n,k+1}$ be the interval with the same left endpoint as $J_{n,k+1}$ with half the length of $J_{n,k+1}$ and $\tilde{J}_{n,m_n+1} = J_{n,m_n+1}$. Then

$$[0, a] = \tilde{J}_{n,1} \cup I_{n,1} \cup \bar{J}_{n,2} \cup \tilde{J}_{n,2} \cup I_{n,2} \cdots \cup \tilde{J}_{n,m_n} \cup I_{n,m_n} \cup \bar{J}_{n,m_n+1}$$

where all I-intervals have the same length, of order $n^{-1/3}\log n$ and the J-intervals have the same length of (smaller) order $n^{-1/3}\sqrt{\log n}$. Finally, let the interval $L_{n,k}$ be defined by

$$L_{n,k} = \tilde{J}_{n,k} \cup I_{n,k} \cup \bar{J}_{n,k+1} = [a_{nk}, a_{n,k+1}), \ \ k = 1, 2, \ldots, m_n, \ \text{yielding } [0, a) = \cup_{k=1}^{m_n} L_{n,k}. \tag{3.29}$$

Note that $m_n \sim an^{1/3}/\log n$ and see the figure below for the structure of the partition.
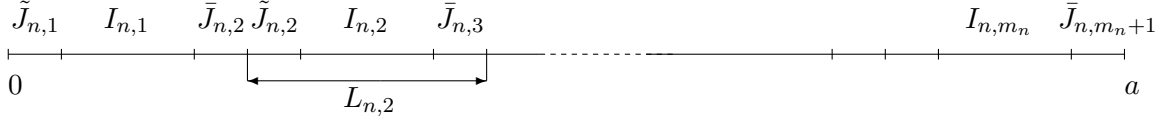
The (key) localization lemma below and proved in the appendix, shows that on intervals $I_{n,k}$ the global convex minorant of V_n (defined in (3.23)) over $[0, a]$ coincides with high probability with the restriction to I_{nk} of the local convex minorant of the process V_n on the interval $L_{n,k}$.

Lemma 3.4 *Let h_0 be strictly positive on $[0, a]$, with a strictly positive continuous derivative h'_0 on $(0, a)$, which also has a strictly positive right limit at 0 and a strictly positive left limit at a. Then:*

(i) *The probability that there exists a k, $1 \le k \le m_n$, such that the greatest convex minorant C_n of V_n is different on the interval I_{nk} from the restriction to I_{nk} of the (local) greatest convex minorant of V_n on L_{nk}, is bounded above by*

$$c_1 \exp\left\{-c_2 (\log n)^{3/2}\right\},$$

for constants $c_1, c_2 > 0$, uniformly in n.

(ii) *The probability that there exists a k, $1 \le k \le m_n$, such that C_n has no change of slope in an interval $\bar{J}_{nk}$ or $\tilde{J}_{nk}$ is also bounded by*

$$c_1 \exp\left\{-c_2 (\log n)^{3/2}\right\},$$

for constants c_1, c_2, uniformly in n.

For each $n \ge 1$ and $1 \le k \le m_n$ define independent standard Brownian motions $W_{n1}, \ldots, W_{n,m_n}$ and consider the processes

$$x \mapsto H_0(x) - H_0(a_{nk}) + n^{-1/2} W_{nk}\left(\frac{F_0(x)}{1 - F_0(x)} - \frac{F_0(a_{nk})}{1 - F_0(a_{nk})}\right), \ x \in L_{nk}.$$

Denote the greatest convex minorants of these processes (on L_{nk}) by C_{nk}. Furthermore, define the processes S_{nk} by

$$S_{nk}(x) = H_0(x) - H_0(a_{nk}) + n^{-1/2} W_{nk}\left(\frac{F_0(x)}{1 - F_0(x)} - \frac{F_0(a_{nk})}{1 - F_0(a_{nk})}\right) - C_{nk}(x), \ x \in L_{nk}. \quad (3.30)$$

Lemma 3.5 *Assume that the conditions of Theorem 3.1 are satisfied. Moreover, let $C(0)$ be defined as in Theorem 2.1 and $\sigma_{H_0}^2$ as in Theorem 3.1. Then:*

$$n^{5/6} \sum_{k=1}^{m_n} \int_{I_{nk}} \{S_{nk}(x) - ES_{nk}(x)\} \, dF_0(x) \xrightarrow{\mathcal{D}} N(0, \sigma_{H_0}^2), \ n \to \infty,$$

where (see (3.27))

$$n^{2/3} \sum_{k=1}^{m_n} \int_{I_{nk}} ES_{nk}(x) \, dF_0(x) \to E|C(0)| \int_0^a \left(\frac{2h_0(t)f_0(t)}{h'_0(t)}\right)^{1/3} dH_0(t), \ n \to \infty, \quad (3.31)$$

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Proof. Let $c_n = n^{1/3}|L_{nk}| \sim \log n$ and $I_{nk} = [a_{nk} + n^{-1/3}\sqrt{c_n}, a_{nk} + n^{-1/3}(c_n - \sqrt{c_n})]$. We then have:

$$n \int_{I_{nk}} \{S_{nk}(x) - ES_{nk}(x)\} \, dF_0(x)$$

$$= \int_{\sqrt{c_n}}^{c_n - \sqrt{c_n}} \left\{ n^{1/6} W_{nk} \left(\frac{F_0(a_{nk} + n^{-1/3}x)}{1 - F_0(a_{nk} + n^{-1/3}x)} - \frac{F_0(a_{nk})}{1 - F_0(a_{nk})} \right) - n^{2/3} C_{nk}(a_{nk} + n^{-1/3}x) \right.$$

$$\left. - E\left\{ n^{1/6} W_{nk} \left(\frac{F_0(a_{nk} + n^{-1/3}x)}{1 - F_0(a_{nk} + n^{-1/3}x)} - \frac{F_0(a_{nk})}{1 - F_0(a_{nk})} \right) - n^{2/3} C_{nk}(a_{nk} + n^{-1/3}x) \right\} \right\}$$

$$\cdot f_0\left(a_{nk} + n^{-1/3}x \right) \, dx,$$

Here we use that the (first two) deterministic terms in (3.30) drop out because of subtraction of the expectation. This implies that

$$n \int_{I_{nk}} \{S_{nk}(x) - ES_{nk}(x)\} \, dF_0(x)$$

$$\overset{\mathcal{D}}{=} \int_{\sqrt{c_n}}^{c_n - \sqrt{c_n}} \left\{ n^{1/6} W \left(\frac{F_0(a_{nk} + n^{-1/3}x)}{1 - F_0(a_{nk} + n^{-1/3}x)} - \frac{F_0(a_{nk})}{1 - F_0(a_{nk})} \right) - C'_{nk}(x) \right.$$

$$\left. - E\left\{ n^{1/6} W \left(\frac{F_0(a_{nk} + n^{-1/3}x)}{1 - F_0(a_{nk} + n^{-1/3}x)} - \frac{F_0(a_{nk})}{1 - F_0(a_{nk})} \right) - C'_{nk}(x) \right\} \right\} \cdot f_0\left(a_{nk} + n^{-1/3}x \right) \, dx,$$

where C'_{nk} is the greatest convex minorant of the process

$$x \mapsto n^{2/3} \left\{ H_0(a_{nk} + n^{-1/3}x) - H_0(a_{nk}) - n^{-1/3}x h_0(a_{nk}) \right\}$$

$$+ n^{1/6} W \left(\frac{F_0(a_{nk} + n^{-1/3}x)}{1 - F_0(a_{nk} + n^{-1/3}x)} - \frac{F_0(a_{nk})}{1 - F_0(a_{nk})} \right), \ x \in [0, c_n].$$

Here we use that adding a linear function to a function does not change the difference between this function and its greatest convex minorant. Note that the integrals on I_{nk} only depend on the increments of the Brownian motion process on the corresponding disjoint intervals L_{nk} and therefore are independent. For the individual integrals we are close to the situation of Theorem 2.3, with, for $c_n \to \infty$, on $[0, c_n]$ (note that n is determined by c_n, $n = e^{c_n}$)

$$F_{c_n}(x) = n^{1/3} \left\{ F_0(a_{nk} + n^{-1/3}x) - F_0(a_{nk}) \right\} = f_0(a_{nk})x(1 + o(1)),$$

$$H_{c_n}(x) = n^{2/3} \left\{ H_0(a_{nk} + n^{-1/3}x) - H_0(a_{nk}) - n^{-1/3}x h_0(a_{nk}) \right\} = \tfrac{1}{2} h'_0(a_{nk})x^2(1 + o(1))$$

$$\text{and } G_{c_n}(x) = n^{1/3} \left\{ \frac{F_0(a_{nk} + n^{-1/3}x)}{1 - F_0(a_{nk} + n^{-1/3}x)} - \frac{F_0(a_{nk})}{1 - F_0(a_{nk})} \right\} = \frac{f_0(a_{nk})x}{(1 - F_0(a_{nk}))^2}(1 + o(1))$$

This yields:

$$\text{var}\left(\frac{n}{\sqrt{c_n}} \int_{I_{nk}} S_{nk}(x) \, dF_0(x) \right) \sim \sigma_{nk}^2, \ n \to \infty,$$

uniformly in $k = 1, \ldots, m_n$, where

$$\sigma_{nk}^2 = \frac{\left(f_0(a_{nk})/\{1 - F_0(a_{nk})\}^2\right)^{5/3} f_0(a_{nk})^2}{\left(\frac{1}{2}h_0'(a_{nk})\right)^{4/3}}\sigma^2$$

$$= \frac{(h_0(a_{nk}))^{10/3} f_0(a_{nk})^{1/3}}{\left(\frac{1}{2}h_0'(a_{nk})\right)^{4/3}}\sigma^2 = \frac{2^{4/3}h_0(a_{nk})^3 \{h_0(a_{nk})f_0(a_{nk})\}^{1/3}}{h_0'(a_{nk})^{4/3}}\sigma^2,$$

and σ^2 is defined as in Theorem 2.1. Likewise, also with $C(0)$ as defined in Theorem 2.1,

$$\frac{n}{c_n}\int_{I_{nk}} ES_{nk}(x)\,dF_0(x) \sim \frac{2^{1/3}h_0(a_{nk})\{h_0(a_{nk})f_0(a_{nk})\}^{1/3}\,E|C(0)|}{h_0'(a_{nk})^{1/3}}$$

Since the fourth moments of

$$\frac{n}{\sqrt{c_n}}\int_{I_{nk}} \{S_{nk}(x) - ES_{nk}(x)\}\,dF_0(x)$$

are uniformly bounded by Theorem 2.3, we get for each $\varepsilon > 0$, using Chebyshev's inequality,

$$\sum_{k=1}^{m_n}\mathbb{P}\left\{m_n^{-1/2}\left|\frac{n}{\sqrt{c_n}}\int_{I_{nk}}\{S_{nk}(x) - ES_{nk}(x)\}\,dF_0(x)\right| \geq \varepsilon\right\} \to 0,\ n \to \infty.$$

Using that $m_n^{-1} \sim a^{-1}n^{-1/3}\log n$ and that the intervals I_{nk} have lengths of order $n^{-1/3}\log n$, we get:

$$m_n^{-1}\sum_{k=1}^{m_n}\sigma_{nk}^2 \sim m_n^{-1}\sum_{k=1}^{m_n}\frac{2^{4/3}h_0(a_{nk})^3\{h_0(a_{nk})f_0(a_{nk})\}^{1/3}}{h_0'(a_{nk})^{4/3}}\sigma^2$$

$$\longrightarrow \frac{2^{4/3}\sigma^2}{a}\int_0^a \frac{h_0(t)^3\{h_0(t)f_0(t)\}^{1/3}}{h_0'(t)^{4/3}}\,dt.$$

Since $m_n = an^{1/3}/c_n$, the normal convergence criterion on p. 316 of LOÈVE (1963) now gives:

$$n^{5/6}\sum_{k=1}^{m_n}\int_{I_{nk}}\{S_{nk}(x) - ES_{nk}(x)\}\,dF_0(x)$$

$$= m_n^{-1/2}\sum_{k=1}^{m_n}\frac{n\sqrt{a}}{\sqrt{c_n}}\int_{I_{nk}}\{S_{nk}(x) - ES_{nk}(x)\}\,dF_0(x) \xrightarrow{\mathcal{D}} N\left(0, \sigma_{H_0}^2\right).$$

Also note that:

$$m_n^{-1/2}\sum_{k=1}^{m_n}\frac{n}{c_n^{1/2}}\int_{I_{nk}} ES_{nk}(x)\,dF_0(x) \sim m_n^{-1/2}c_n^{1/2}\sum_{k=1}^{m_n}\frac{2^{1/3}h_0(a_{nk})\{h_0(a_{nk})f_0(a_{nk})\}^{1/3}}{h_0'(a_{nk})^{1/3}}$$

$$\sim \sqrt{m_n c_n}E|C(0)|\int_0^a \frac{2^{1/3}h_0(t)\{h_0(t)f_0(t)\}^{1/3}}{h_0'(t)^{1/3}}\,dt = n^{1/6}\int_0^a\left(\frac{2h_0(t)f_0(t)}{h_0'(t)}\right)^{1/3}\,dH_0(t).$$

$\square$

In applications of this result, used in a bootstrap approach to the computation of critical values, we need the following lemma, which gives a more precise expansion of the asymptotic representation of the expectation, given in (3.31).

Lemma 3.6 *Assume that the conditions of Theorem 3.1 are satisfied and assume in addition that h_0 has a bounded second derivative on $[0, a]$. Then*

$$n^{2/3} E D_n = E|C(0)| \int_0^a \left(\frac{2 h_0(t) f_0(t)}{h_0'(t)} \right)^{1/3} dH_0(t) + o\left(n^{-1/6} \right). \tag{3.32}$$

Proof. We have:

$$\frac{n}{c_n} \int_{I_{nk}} E S_{nk}(x) \, dF_0(x) = \frac{h_0(a_{nk}) \{ 2 h_0(a_{nk}) f_0(a_{nk}) \}^{1/3} E|\zeta(0)|}{h_0'(a_{nk})^{1/3}} + o\left(n^{-1/6} \right), \tag{3.33}$$

uniformly in $k = 1, \ldots, m_n$. This is seen in the following way.

On the intervals I_{nk} we get:

$$F_{c_n}(x) \overset{\text{def}}{=} n^{1/3} \left\{ F_0(a_{nk} + n^{-1/3} x) - F_0(a_{nk}) \right\} = f_0(a_{nk}) x + O\left(n^{-1/3} (\log n)^2 \right),$$

$$H_{c_n}(x) \overset{\text{def}}{=} n^{2/3} \left\{ H_0(a_{nk} + n^{-1/3} x) - H_0(a_{nk}) - n^{-1/3} x h_0(a_{nk}) \right\}$$

$$= \tfrac{1}{2} h_n'(a_{nk}) x^2 + O\left(n^{-1/3} (\log n)^3 \right),$$

$$G_{c_n}(x) \overset{\text{def}}{=} n^{1/3} \left\{ \frac{F_0(a_{nk} + n^{-1/3} x)}{1 - F_0(a_{nk} + n^{-1/3} x)} - \frac{F_0(a_{nk})}{1 - F_0(a_{nk})} \right\} = \frac{f_0(a_{nk}) x}{(1 - F_0(a_{nk}))^2} + O\left(n^{-1/3} (\log n)^2 \right).$$

uniformly in $k = 1, \ldots, m_n$. The relation for F_{c_n} and G_{c_n} immediately follow from the mean value theorem, applied on the remainder term, together with the conditions of Theorem 3.1, which yield that h_0' and f_0' are uniformly bounded. In the expansion of H_{c_n} we use the boundedness of the second derivative h_0''.

Combining these relations gives (3.33), and hence:

$$n^{2/3} \sum_{k=1}^{m_n} \int_{I_{nk}} E S_{nk}(x) \, dF_0(x) = \frac{n}{c_n} \sum_{k=1}^{m_n} \int_{I_{nk}} E S_{nk}(x) \, dF_0(x) \, |I_{nk}|$$

$$= \frac{n}{c_n} \sum_{k=1}^{m_n} \frac{h_0(a_{nk}) \{ 2 h_0(a_{nk}) f_0(a_{nk}) \}^{1/3} E|\zeta(0)|}{h_0'(a_{nk})^{1/3}} |I_{nk}| + O\left(n^{-1/3} (\log n)^3 \sum_{k=1}^{m_n} |I_{nk}| \right)$$

$$= E|C(0)| \sum_{k=1}^{m_n} \int_{I_{nk}} \left(\frac{2 h_0(t) f_0(t)}{h_0'(t)} \right)^{1/3} dH_0(t) + O\left(n^{-1/3} (\log n)^3 \right)$$

where, in the last line, we use again the boundedness of f_0', h_0' and h_0'', combined with the mean value theorem on the intervals I_{nk}. Note that part (i) of Lemma 3.4 tells us that the probability that S_n is different from S_{nk} on I_{nk} is bounded above by

$$c_1 \exp \left\{ -c_2 (\log n)^{3/2} \right\},$$

so we also have:

$$n^{2/3} \sum_{k=1}^{m_n} \int_{I_{nk}} E S_n(x) \, dF_0(x) = E|C(0)| \sum_{k=1}^{m_n} \int_{I_{nk}} \left(\frac{2 h_0(t) f_0(t)}{h_0'(t)} \right)^{1/3} dH_0(t) + O\left(n^{-1/3} (\log n)^3 \right).$$

It is clear that we get in a similar way:

$$n^{2/3} \sum_{k=1}^{m_n} \int_{L_{nk} \setminus I_{nk}} ES_n(x)\, dF_0(x)$$

$$= E|C(0)| \sum_{k=1}^{m_n} \int_{L_{nk} \setminus I_{nk}} \left(\frac{2h_0(t) f_0(t)}{h_0'(t)} \right)^{1/3} dH_0(t) + O\left(n^{-1/3} (\log n)^{3/2} \right).$$

Hence

$$n^{2/3} ED_n = n^{2/3} \sum_{k=1}^{m_n} \int_{L_{nk}} ES_n(x)\, dF_0(x) = \int_0^a \left(\frac{2h_0(t) f_0(t)}{h_0'(t)} \right)^{1/3} dH_0(t) + o\left(n^{-1/6} \right).$$

$\square$

We can now prove Theorem 3.1.

Proof of Theorem 3.1. By Lemmas 3.4 and 3.5 we have:

$$n^{5/6} \sum_{k=1}^{m_n} \int_{I_{nk}} \{S_{nk}(x) - ES_{nk}(x)\}\, dF_0(x) \xrightarrow{\mathcal{D}} N(0, \sigma_{H_0}^2), \ n \to \infty.$$

By part (i) of Lemma 3.4, the probability that S_n is different from S_{nk} on I_{nk} is bounded above by

$$c_1 \exp\left\{ -c_2 (\log n)^{3/2} \right\},$$

implying that also:

$$n^{5/6} \sum_{k=1}^{m_n} \int_{I_{nk}} \{S_n(x) - ES_n(x)\}\, dF_0(x) \xrightarrow{\mathcal{D}} N(0, \sigma_{H_0}^2), \ n \to \infty.$$

For similar reasons we have:

$$n^{5/6} \sum_{k=1}^{m_n} \int_{L_{nk} \setminus I_{nk}} \{S_n(x) - ES_n(x)\}\, dF_0(x) \xrightarrow{p} 0, \ n \to \infty,$$

where we use Theorem 2.3 (this is the essence of the "big blocks, small blocks" method). The result now follows, since

$$D_n = n^{5/6} \sum_{k=1}^{m_n} \int_{L_{nk}} \{S_n(x) - ES_n(x)\}\, dF_0(x).$$

$\square$

4 Further central limit results

In order to derive the asymptotic distribution of the statistic U_n defined in (1.2) and used in the simulations in GROENEBOOM AND JONGBLOED (2011A), we first consider the statistic

$$\int_{[0,a]} \left\{ \mathbb{F}_n(x) - \hat{F}_n(x) \right\} dF_0(x),$$

which is analogous to the statistic discussed in the preceding section, but has $\mathbb{F}_n(x) - \hat{F}_n(x)$ as integrand instead of $\mathbb{H}_n(x) - \hat{H}_n(x)$. We have, if E_n again denotes the empirical process, and arguing as in the proof of Lemma 3.2 (i),

$$\mathbb{F}_n(x) = 1 - \exp\left\{-\mathbb{H}_n(x)\right\} = 1 - \exp\left\{-H_0(x) + \log\left\{1 - \frac{n^{-1/2}E_n(x)}{1 - F_0(x)}\right\}\right\}$$

$$= 1 - \exp\left\{-H_0(x) - \frac{n^{-1/2}E_n(x)}{1 - F_0(x)}\right\} + O_p\left(n^{-1}\right),$$

uniformly for $x \in [0, a]$. Hence, defining, as in Lemma 3.2, H_n as the greatest convex minorant of the process

$$x \mapsto H_0(x) + \frac{n^{-1/2}E_n(x)}{1 - F_0(x)},$$

we get, by Lemma 3.2,

$$\mathbb{F}_n(x) - \hat{F}_n(x) = \exp\left\{-H_n(x)\right\} - \exp\left\{-H_0(x) - \frac{n^{-1/2}E_n(x)}{1 - F_0(x)}\right\} + O_p\left(n^{-1}\right)$$

$$= \exp\left\{-H_n(x)\right\}\left\{1 - \exp\left\{-H_0(x) - \frac{n^{-1/2}E_n(x)}{1 - F_0(x)} + H_n(x)\right\}\right\} + O_p\left(n^{-1}\right).$$

Next, replacing $E_n(x)$ by $B_n(F_0(x))$, as in Lemma 3.3, where (B_n) are the approximating Brownian bridges, we get:

$$\mathbb{F}_n(x) - \hat{F}_n(x)$$
$$= \exp\left\{-C_n^B(x)\right\}\left\{1 - \exp\left\{-H_0(x) - \frac{n^{-1/2}B_n(F_0(x))}{1 - F_0(x)} + C_n^B(x)\right\}\right\} + O_p\left(\frac{\log n}{n}\right),$$

where C_n^B is the greatest convex minorant of the process

$$x \mapsto H_0(x) + \frac{n^{-1/2}B_n(F_0(x))}{1 - F_0(x)}, \ x \in [0, a].$$

Again using the results of the preceding section, it is seen that this implies that

$$\mathbb{F}_n(x) - \hat{F}_n(x)$$
$$\overset{\mathcal{D}}{=} \exp\left\{-C_n(x)\right\}\left\{1 - \exp\left\{-H_0(x) - n^{-1/2}W\left(\frac{F_0(x)}{1 - F_0(x)}\right) + C_n(x)\right\}\right\} + O_p\left(\frac{\log n}{n}\right), \quad (4.34)$$

where W is standard Brownian motion on $[0, \infty)$, and C_n is the greatest convex minorant of the process

$$x \mapsto H_0(x) + n^{-1/2}W\left(\frac{F_0(x)}{1 - F_0(x)}\right), \ x \in [0, a].$$

This representation suggests to consider

$$\exp\left\{-H_0(x)\right\}\left\{H_0(x) + n^{-1/2}W\left(\frac{F_0(x)}{1 - F_0(x)}\right) - C_n(x)\right\}$$

$$= \{1 - F_0(x)\}\left\{H_0(x) + n^{-1/2}W\left(\frac{F_0(x)}{1 - F_0(x)}\right) - C_n(x)\right\}, \ x \in [0, a].$$

We have the following result.

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Lemma 4.1 *Let h_0 be strictly positive on $[0, a]$, with a strictly positive continuous derivative h_0' on $(0, a)$, which also has a strictly positive right limit at 0 and a strictly positive left limit at a. Moreover, let C_n, S_n and V_n be defined as in Theorem 3.1 and let $D_n^{F_0}$ be defined by*

$$D_n^{F_0} = \int_0^a S_n(x)\{1 - F_0(x)\}\, dF_0(x), \tag{4.35}$$

Then:

$$n^{5/6}\left\{D_n^{F_0} - ED_n^{F_0}\right\} \xrightarrow{\mathcal{D}} N(0, \sigma_{F_0}^2), \ n \to \infty,$$

where

$$ED_N^{F_0} \sim n^{-2/3} E|C(0)| \int_0^a \left(\frac{2h_0(t)f_0(t)}{h_0'(t)}\right)^{1/3} dF_0(t), \ n \to \infty,$$

$$\sigma_{F_0}^2 = \sigma^2 \int_0^a \left(\frac{2h_0(t)f_0(t)}{h_0'(t)}\right)^{4/3} dF_0(t),$$

and σ^2 is defined as in Theorem 2.1.

Proof. The only difference with Theorem 3.1 is that $dF_0(t)$ is replaced by $\{1 - F_0(t)\}\, dF_0(t)$ in the integral. This means that instead of ED_n we get:

$$ED_N^{F_0} \sim n^{-2/3} E|C(0)| \int_0^a \frac{2^{1/3}h_0(t)\{h_0(t)f_0(t)\}^{1/3}\{1 - F_0(t)\}}{h_0'(t)^{1/3}} dt, \ n \to \infty,$$

and instead of $\sigma_{H_0}^2$ we get:

$$\sigma_{F_0}^2 = 2^{4/3}\sigma^2 \int_0^a \frac{h_0(t)^3\{h_0(t)f_0(t)\}^{1/3}}{h_0'(t)^{4/3}}\{1 - F_0(t)\}^2 dt = \sigma^2 \int_0^a \left(\frac{2h_0(t)f_0(t)}{h_0'(t)}\right)^{4/3} dF_0(t).$$

$\square$

We have the following corollary.

Corollary 4.1 *Let h_0 be strictly positive on $[0, a]$, with a strictly positive continuous derivative h_0' on $(0, a)$, which also has a strictly positive right limit at 0 and a strictly positive left limit at a. Moreover, let S_n' be defined by:*

$$S_n'(x) = \mathbb{F}_n(x) - \hat{F}_n(x), \ x \in [0, a],$$

where $\hat{F}_n$ is defined in (1.2) and let D_n' be defined by

$$D_n' = \int_0^a S_n'(x)\, dF_0(x), \tag{4.36}$$

Then,

$$n^{5/6}\left\{D_n' - ED_n'\right\} \xrightarrow{\mathcal{D}} N(0, \sigma_{F_0}^2), \ n \to \infty,$$

where $\sigma_{F_0}^2$ is defined as in Lemma 4.1.

Proof. This is (in a sense) an application of the delta method. By (4.34) we can replace $\mathbb{F}_n - \hat{F}_n$ by:

$$\exp\left\{-C_n(x)\right\}\left\{1 - \exp\left\{-H_0(x) - n^{-1/2}W\left(\frac{F_0(x))}{1 - F_0(x)}\right) + C_n(x)\right\}\right\}.$$

We also have, using notation of the same type as in the proof of Lemma 3.5,

$$\int_0^a E\left\{H_0(x) - C_n(x)\right\}^2 dF_0(x)$$

$$\sim \sum_{k=1}^{m_n} \int_0^{c_n} E\left\{H_0(a_{nk} + n^{-1/3}u) - H_0(a_{nk}) - C_n(a_{nk} + n^{-1/3}u) + C_n(a_{nk})\right\}^2 f_0(a_{nk})\,du$$

$$\sim n^{-5/3} \sum_{k=1}^{m_n} \int_0^{c_n} E\left\{\tfrac{1}{2}h_0'(a_{nk})u^2 - C_{nk}(u)\right\}^2 f_0(a_{nk})\,du,$$

where C_{nk} is the greatest convex minorant of the process

$$x \mapsto \tfrac{1}{2}h_0'(a_{nk})u^2 + W\left(\frac{h_0(a_{nk})u}{1 - F_0(a_{nk})}\right), \, u \in [0, c_n].$$

By Brownian scaling, we get:

$$\int_0^{c_n} E\left\{\tfrac{1}{2}h_0'(a_{nk})u^2 - C_{nk}(u)\right\}^2 f_0(a_{nk})\,du$$

$$\sim c_n f_0(a_{nk}) \left(\tfrac{1}{2}h_0'(a_{nk})\right)^{2/3} \left(\frac{h_0(a_{nk})}{1 - F_0(a_{nk})}\right)^{4/3} EC(0)^2,$$

where C is the greatest convex minorant of $x \mapsto W(x) + x^2$, $x \in \mathbb{R}$. So we find:

$$\int_0^a E\left\{H_0(x) - C_n(x)\right\}^2 dF_0(x) \sim n^{-4/3} EC(0)^2 \int_0^a \left(\tfrac{1}{2}h_0'(t)\right)^{2/3} \left(\frac{h_0(t)}{1 - F_0(t)}\right)^{4/3} dF_0(t). \quad (4.37)$$

We also have:

$$\int_0^a E\left\{H_0(x) + n^{-1/2}W\left(\frac{F_0(x)}{1 - F_0(x)}\right) - C_n(x)\right\}^2 dF_0(x)$$

$$\sim n^{-4/3} EC(0)^2 \int_0^a \left(\tfrac{1}{2}h_0'(t)\right)^{2/3} \left(\frac{h_0(t)}{1 - F_0(t)}\right)^{4/3} dF_0(t). \quad (4.38)$$

Hence, by (4.37) and (4.38),

$$\int_0^a \exp\left\{-C_n(x)\right\}\left\{1 - \exp\left\{-H_0(x) - n^{-1/2}W\left(\frac{F_0(x))}{1 - F_0(x)}\right) + C_n(x)\right\}\right\} dF_0(t)$$

$$= \int_0^a \left\{1 - F_0(t)\right\}\left\{1 - \exp\left\{-H_0(x) - n^{-1/2}W\left(\frac{F_0(x))}{1 - F_0(x)}\right) + C_n(x)\right\}\right\} dF_0(t) + O_p\left(n^{-4/3}\right)$$

$$= \int_0^a \left\{1 - F_0(t)\right\}\left\{H_0(x) + n^{-1/2}W\left(\frac{F_0(x))}{1 - F_0(x)}\right) - C_n(x)\right\} dF_0(t) + O_p\left(n^{-4/3}\right),$$

where we also use the Cauchy-Schwarz inequality in the first equality.

For the expectation we get similarly

$$\int_0^a E\left\{\mathbb{F}_n(x) - \hat{F}_n(x)\right\} dF_0(x) = ED_n^{F_0} + O\left(\frac{\log n}{n}\right),$$

where $D_n^{F_0}$ is defined by (4.35). This is seen in the following way. Since we assume that $F_0(a) < 1$, we have by by Chernoff's theorem (as in the proof of Lemma 3.2),

$$\mathbb{P}\left\{1 - \mathbb{F}_n(a) < \tfrac{1}{2}\{1 - F_0(a)\}\right\} \leq e^{-nc},$$

for a $c > 0$, and hence, defining the event A_n by

$$A_n = \left\{1 - \mathbb{F}_n(a) \geq \tfrac{1}{2}\{1 - F_0(a)\}\right\},$$

we get

$$\int_0^a E\left\{\mathbb{F}_n(x) - \hat{F}_n(x)\right\} dF_0(x) = \int_0^a E\left\{\mathbb{F}_n(x) - \hat{F}_n(x)\right\} 1_{A_n} dF_0(x) + O\left(e^{-nc}\right)$$

$$= \int_0^a E\left\{e^{-\hat{H}_n(x)} - e^{-\mathbb{H}_n(x)}\right\} 1_{A_n} dF_0(x) + O\left(e^{-nc}\right)$$

$$= \int_0^a E\{1 - \mathbb{F}_n(x)\}\left\{e^{-\left\{\hat{H}_n(x) - \mathbb{H}_n(x)\right\}} - 1\right\} 1_{A_n} dF_0(x) + O\left(e^{-nc}\right)$$

$$= \int_0^a \{1 - F_0(x)\} E\left\{e^{-\left\{\hat{H}_n(x) - \mathbb{H}_n(x)\right\}} - 1\right\} 1_{A_n} dF_0(x) + O\left(n^{-1}\right)$$

$$= \int_0^a \{1 - F_0(x)\} E\left\{H_0(x) + n^{-1/2} W\left(\frac{F_0(x))}{1 - F_0(x)}\right) - C_n(x)\right\} dF_0(x) + O\left(\frac{\log n}{n}\right)$$

$$= ED_n^{F_0} + O\left(\frac{\log n}{n}\right).$$

The result now follows from Lemma 4.1. □

Similarly as in Lemma 3.6, we have the following expansion for the expectation in Corollary 4.1. The proof proceeds along the same lines as the proof of 3.6 and is therefore omitted.

Lemma 4.2 *Assume that the conditions of Corollary 4.1 are satisfied and assume in addition that h_0 has a bounded second derivative on $[0, a]$. Then*

$$n^{2/3} ED_n' = E|C(0)| \int_0^a \left(\frac{2h_0(t) f_0(t)}{h_0'(t)}\right)^{1/3} dF_0(t) + o\left(n^{-1/6}\right). \tag{4.39}$$

The preceding results finally yield the following theorems.

Theorem 4.1 *Let D_n be defined as in Theorem 3.1 and let the conditions of Theorem 3.1 be satisfied. Then:*

$$n^{5/6} \left\{\int_0^a \left\{\mathbb{H}_n(x-) - \hat{H}_n(x)\right\} d\mathbb{F}_n(x) - ED_n\right\} \xrightarrow{\mathcal{D}} N(0, \sigma_{H_0}^2), \ n \to \infty,$$

where ED_n and $\sigma_{H_0}^2$ are defined as in Theorem 3.1.

Theorem 4.2 *Let the conditions of Lemma 4.1 be satisfied and let $\sigma_{F_0}^2$ and $C(0)$ be defined as in Lemma 4.1. Then:*

$$n^{5/6} \left\{ \int_0^a \left\{ \mathbb{F}_n(x-) - \hat{F}_n(x) \right\} d\mathbb{F}_n(x) - \int_0^a E\left\{ \mathbb{F}_n(x-) - \hat{F}_n(x) \right\} dF_0(x) \right\} \xrightarrow{\mathcal{D}} N(0, \sigma_{F_0}^2), \ n \to \infty,$$

and

$$\int_0^a E\left\{ \mathbb{F}_n(x-) - \hat{F}_n(x) \right\} dF_0(x) \sim n^{-2/3} E|C(0)| \int_0^a \left(\frac{2h_0(t)f_0(t)}{h_0'(t)} \right)^{1/3} dF_0(t), \ n \to \infty.$$

We only prove Theorem 4.1, since the proof of Theorem 4.2 proceeds along similar lines.

Proof of Theorem 4.1. Using Lemma 3.1 again, we get:

$$\int_0^a \left\{ \mathbb{H}_n(x-) - \hat{H}_n(x) \right\} d\mathbb{F}_n(x) \stackrel{\mathcal{D}}{=} \int_0^a \left\{ H_0(x) + n^{-1/2} W_n \left(\frac{F_0(x)}{1 - F_0(x)} \right) - C_n(x) \right\} d\mathbb{F}_n(x) + O_p\left(\frac{\log n}{n} \right),$$

where C_n is the greatest convex minorant of V_n which is defined as in (3.23) with W_n replacing W. The process W_n is distributed as standard Brownian motion on $[0, a]$ and $W_n \circ (F_0/(1 - F_0))$ is given by

$$W_n \left(\frac{F_0(x)}{1 - F_0(x)} \right) = \frac{B_n(F_0(x))}{1 - F_0(x)}, \ x \in [0, a],$$

where B_n is coupled to the empirical process as in Lemma 3.3.

We only have to show

$$\int_0^a \left\{ V_n(x) - C_n(x) \right\} d\left(\mathbb{F}_n - F_0 \right)(x) = o_p\left(n^{-5/6} \right), \tag{4.40}$$

since we then have:

$$\int_0^a \left\{ \mathbb{H}_n(x-) - \hat{H}_n(x) \right\} d\mathbb{F}_n(x) = \int_0^a \left\{ V_n(x) - C_n(x) \right\} d\mathbb{F}_n(x) + O_p\left(\frac{\log n}{n} \right)$$

$$= \int_0^a \left\{ V_n(x) - C_n(x) \right\} dF_0(x) + \int_0^a \left\{ V_n(x) - C_n(x) \right\} d\left(\mathbb{F}_n - F_0 \right)(x) + O_p\left(\frac{\log n}{n} \right)$$

$$= \int_0^a \left\{ V_n(x) - C_n(x) \right\} dF_0(x) + o_p\left(n^{-5/6} \right).$$

To show that this relation holds, we follow a method which is somewhat similar to the method used in KULIKOV AND LOPUHAÄ (2008) (but uses the Brownian motion representation instead of the empirical process and does not bring the derivative of the greatest convex minorant into play).

The p-variation of a function f on the interval $I = [0, a]$ is defined by

$$\nu_p(f; I) = \sup \left\{ \sum_{i=1}^m |f(x_i) - f(x_{i-1})|^p : x_0 = 0 < x_1 < \cdots < x_m = a \right\}.$$

The p-variation norm of f on I is defined by

$$\|f\|_{[p]} = \nu_p(f; I)^{1/p} + \sup_{x \in I} |f(x)|.$$

We have, by Theorem II.3.27 in DUDLEY AND NORVAIŠA (1999), for $p, q > 0$ and $1/p + 1/q > 1$:

$$\left| \int_{[0,a]} \{V_n(x) - C_n(x)\} \, d\left(\mathbb{F}_n - F_0\right)(x) \right| \leq c \, \|V_n - C_n\|_{[p]} \|\mathbb{F}_n - F_0\|_{[q]}, \tag{4.41}$$

for a constant $c > 0$. Moreover, by Theorems I.6.1 and I.6.2 in DUDLEY AND NORVAIŠA (1999), and Theorem 3.2 in QIAN (1998), we have:

$$\|\mathbb{F}_n - F_0\|_{[q]} = \begin{cases} O_p\left(n^{(1-q)/q}\right) & , q \in [1, 2), \\ O_p\left(n^{-1/2}\sqrt{L(Ln))}\right) & , q = 2, \\ O_p\left(n^{-1/2}\right) & , q > 2, \end{cases} \tag{4.42}$$

where $Ln = 1 \vee \log n$.

Let $\tau_1, \ldots, \tau_m$ be the points of jump of the derivative c_n of C_n on $[0, a]$, and let $\tau_0 = 0$, $\tau_{m+1} = a$. The function C_n is linear on the intervals $[\tau_i, \tau_{i+1}]$, and V_n behaves on such an interval as an excursion above its greatest convex minorant C_n, with the same values as V_n at the endpoints of the interval. Hence we have, for $p > 2$, by Lemma 4 of HUANG AND DUDLEY (2001),

$$\nu_p(V_n - C_n; [0, a]) \leq 2^{p-1} \sum_{k=1}^{m+1} \nu_p(V_n - C_n; [\tau_{i-1}, \tau_i]) = 2^{p-1} \sum_{i=1}^{m+1} \nu_p(V_n; [\tau_{i-1}, \tau_i]),$$

where, using the fact that the linear part drops out in taking the comparison with the greatest convex minorant,

$$\tilde{V}_n(x) = n^{-1/2} \left\{ W\left(\frac{F_0(x)}{1 - F_0(x)} - \frac{F_0(\tau_{i-1})}{1 - F_0(\tau_{i-1})} \right) - \frac{x - \tau_{i-1}}{\tau_i - \tau_{i-1}} W\left(\frac{F_0(\tau_i)}{1 - F_0(\tau_i)} - \frac{F_0(\tau_{i-1})}{1 - F_0(\tau_{i-1})} \right) \right\}$$
$$+ H_0(x) - H_0(\tau_{i-1}) - \frac{x - \tau_{i-1}}{\tau_i - \tau_{i-1}} \{H_0(\tau_i) - H_0(\tau_{i-1})\}, \, x \in [\tau_{i-1}, \tau_i].$$

By part (ii) of Lemma 3.4 we have:

$$E \max_i (\tau_i - \tau_{i-1}) = O\left(n^{-1/3} \log n\right).$$

Let u_i be the midpoint of the interval $[\tau_{i-1}, \tau_i]$ and let f_{H_0} by defined by

$$f_{H_0}(x) = H_0(x) - H_0(\tau_{i-1}) - \frac{x - \tau_{i-1}}{\tau_i - \tau_{i-1}} \{H_0(\tau_i) - H_0(\tau_{i-1})\}, \, x \in [\tau_{i-1}, \tau_i].$$

Then

$$f_{H_0}(x) = -\tfrac{1}{2} h_0'(u_i)\{x - \tau_{i-1}\}\{\tau_i - x\}\{1 + o_p(1)\},$$

where $x \mapsto \{x - \tau_{i-1}\}\{\tau_i - x\}$ is increasing on $[\tau_{i-1}, u_i]$ and decreasing on $[u_i, \tau_i]$, and

$$\nu_p\left(f_{H_0}; [\tau_{i-1}, \tau_i]\right)) \sim 2^{1-p} h_0'(u_i)^p \{u_i - \tau_{i-1})\}^p \{\tau_i - u_i)\}^p,$$

(see, e.g., (3.4) of HUANG AND DUDLEY (2001)). Hence, for any $p > 2$,

$$\sum_{i=1}^{m+1} \nu_p(f_{H_0}; [\tau_{i-1}, \tau_i])$$

$$\sim 2^{1-p} \sum_{i=1}^{m+1} h_0'(u_i)^p \{u_i - \tau_{i-1}\}^p \{\tau_i - u_i\}^p = 2^{1-3p} \sum_{i=1}^{m+1} h_0'(u_i)^p \{u_i - \tau_{i-1}\}^{2p}$$

$$\leq 2^{-3p} \max_i \{u_i - \tau_{i-1}\}^{2p-1} \sum_{i=1}^{m+1} h_0'(u_i)^p \{\tau_i - \tau_{i-1}\}$$

$$\sim 2^{-5p+1} \max_i \{\tau_i - \tau_{i-1}\}^{2p-1} \int_0^a h_0'(u)^p \, du = O_p\left(n^{-(2p-1)/3}(\log n)^{(2p-1)/2}\right).$$

Note that the O_p-term becomes $O_p\left(n^{-1}(\log n)^{3/2}\right)$ for $p = 2$.

For the Brownian part

$$B_{nk}(x) \stackrel{\text{def}}{=} n^{-1/2} \left\{ W\left(\frac{F_0(x)}{1 - F_0(x)} - \frac{F_0(\tau_{i-1})}{1 - F_0(\tau_{i-1})}\right) - \frac{x - \tau_{i-1}}{\tau_i - \tau_{i-1}} W\left(\frac{F_0(\tau_i)}{1 - F_0(\tau_i)} - \frac{F_0(\tau_{i-1})}{1 - F_0(\tau_{i-1})}\right) \right\},$$

we find, for $p > 2$,

$$\sum_{i=1}^{m+1} \nu_p(B_{nk}; [\tau_{i-1}, \tau_i]) = O_p\left(n^{-p/2}\right),$$

by the fact that almost all Brownian motion paths are Hölder continuous of any order $< 1/2$.

So we find:

$$\|V_n - C_n\|_{[p]} = O_p\left(n^{-1/2}(\log n)^{(2p-1)/(2p)}\right), \tag{4.43}$$

for any $p > 2$. Thus (4.41), (4.42) and (4.43) imply

$$\int_{[0,a]} \{V_n(x) - C_n(x)\} \, d\left(\mathbb{F}_n - F_0\right)(x) = O_p\left(n^{-1+\varepsilon}\right),$$

for arbitrarily small $\varepsilon > 0$. □

We end this section with a result for the situation that the hazard is nondecreasing, but not strictly nondecreasing.

Theorem 4.3 *Let $\hat{F}_n$ and $\mathbb{F}_n$ be defined as in Theorem 4.2 and let (again)*

$$U_n = \int_{[0,a]} \{\mathbb{F}_n(x-) - \hat{F}_n(x)\} \, d\mathbb{F}_n(x).$$

Let U be given by

$$U = \int_0^a \{1 - F_0(x)\} \left\{ W\left(\frac{F_0(x)}{1 - F_0(x)}\right) - C(x) \right\} dF_0(x),$$

where W is standard Brownian motion on $[0, \infty)$ and C is the greatest convex minorant of

$$x \mapsto W\left(\frac{F_0(x)}{1 - F_0(x)}\right), \ x \in [0, a]. \tag{4.44}$$

Suppose that the underlying hazard h_0 is constant on $[0, a]$. Then:

$$n^{1/2} U_n \stackrel{\mathcal{D}}{\longrightarrow} U, \ n \to \infty.$$

Proof. The proof follows lines that are familiar by now. We first consider

$$U'_n = \int_{[0,a]} \left\{ \mathbb{F}_n(x-) - \hat{F}_n(x) \right\} dF_0(x).$$

By (4.34) we can replace $\mathbb{F}_n - \hat{F}_n$ by:

$$\exp\left\{-C_n(x)\right\} \left\{ 1 - \exp\left\{ -H_0(x) - n^{-1/2} W\left(\frac{F_0(x))}{1 - F_0(x)} \right) - C_n(x) \right\} \right\},$$

where C_n is the greatest convex minorant of the process

$$x \mapsto H_0(x) + n^{-1/2} W\left(\frac{F_0(x))}{1 - F_0(x)} \right), \; x \in [0,a],$$

with a remainder term of order $O_p((\log n)/n)$. Using the delta method as in the proof of Corollary 4.1, we can replace this (apart from a remainder term of order $O_p(n^{-1})$) by:

$$n^{-1/2} \{1 - F_0(x)\} \left\{ W\left(\frac{F_0(x))}{1 - F_0(x)} \right) - C(x) \right\}, \; x \in [0,a].$$

where C is the greatest convex minorant of the process (4.44), and where we use that H_0 is linear on $[0,a]$. The statement for U_n now follows by an application of DUDLEY AND NORVAIŠA (1999), as in the proof of Theorem 4.1. □

Remark 4.1 Note that the rate of convergence drops from $n^{5/6}$ to $n^{1/2}$ in Theorem 4.3, and that the limiting distribution is not normal. We get a limit behavior that can be analyzed using the methods of GRONEBOOM (1983), where the concave majorant of Brownian motion without drift is characterized via a Poisson process of jump locations and Brownian excursions.

Appendix

Proof of Lemma 2.1. Let $u > 0$. Then, for $x \geq u$:

$$V(x) = W(x) + (x-u)^2 + 2u(x-u) + u^2 \geq W(x) + (x-u)^2 + u^2 = W(u) + u^2 + W(x) - W(u) + (x-u)^2.$$

Hence,

$$\mathbb{P}\left(\min_{x \geq u} V(x) \leq 0 \right) \leq \mathbb{P}\left(\min_{x \geq u} W(u) + u^2 + W(x) - W(u) + (x-u)^2 \leq 0 \right) =$$

$$\mathbb{P}\left(W(u) + u^2 + \min_{x \geq u} W(x) - W(u) + (x-u)^2 \leq 0 \right) \leq$$

$$\mathbb{P}\left(W(u) \leq -\tfrac{1}{2} u^2 \right) + \mathbb{P}\left(\min_{x \geq u} W(x) - W(u) + (x-u)^2 \leq -\tfrac{1}{2} u^2 \right)$$

The process

$$x \mapsto W(x) - W(u) + (x-u)^2, \; x \geq u$$

behaves in the same way as the process $t \mapsto V(t)$, $t \geq 0$, but starts in x instead of 0. By Corollary 2.1 in GRONEBOOM AND TEMME (2010) we have that for all $z > 0$,

$$\mathbb{P}\left\{ \min_{t \in \mathbb{R}} V(t) \leq -z \right\} \sim 2 \cdot 3^{-1/2} \exp\left\{ -8z^{3/2}/\sqrt{27} \right\}, \; z \to \infty, \tag{4.45}$$

implying that there exist positive constants c_1 and c_2 such that for all $u \geq 0$

$$\mathbb{P}\left(\min_{x \geq u} W(x) - W(u) + (x-u)^2 \leq -\tfrac{1}{2}u^2\right) \leq c_1 \exp\left\{-c_2 u^3\right\}.$$

We also have for all $u > 0$

$$\mathbb{P}\left\{W(u) < -\tfrac{1}{2}u^2\right\} = \mathbb{P}\left\{W(u)/\sqrt{u} < -\tfrac{1}{2}u^{3/2}\right\} \leq \frac{\exp\left\{-\tfrac{1}{8}u^3\right\}}{u^{3/2}\sqrt{\pi/2}}$$

implying that there exist positive constants c_3 and c_4 such that for all $u \geq 0$

$$\mathbb{P}\left\{W(u) < -\tfrac{1}{2}u^2\right\} \leq c_3 \exp\left\{-c_4 u^3\right\}.$$

Combining these upper bounds with the fact that the process V running to the left from zero behaves in the same way as the process V running to the right from zero, the statement of the lemma follows. $\qquad\square$

Proof of Lemma 3.4. (i). The interval $I_{n,k}$ is bounded on the left by the interval $\tilde{J}_{n,k}$ and on the right by the interval $\bar{J}_{n,k+1}$. The intervals $\tilde{J}_{n,k}$ and $\bar{J}_{n,k+1}$ both have length of order $n^{-1/3}\sqrt{\log n}$. If the greatest convex minorant C_n of V_n on $[0,a]$ has changes of slope in the intervals $\tilde{J}_{n,k}$ and $\bar{J}_{n,k+1}$, the greatest convex minorant of V_n on $[0,a]$, restricted to the interval $I_{n,k}$, coincides with the greatest convex minorant C_{nk} of V_n on $L_{n,k}$, restricted to the interval $I_{n,k}$. So we have to find bounds for the probability that the greatest convex minorant of V_n on $[0,a]$ has no changes of slope in $\tilde{J}_{nk}$ or $\bar{J}_{n,k+1}$. To do this, we follow the method used in GROENEBOOM AND WELLNER (1992), p. 96.

Let a_{nk} and b_{nk} be the left and right endpoints of $\bar{J}_{n,k+1}$, respectively, and let u_{nk} be its midpoint. If

$$c_n(a_{nk}) < h_0(u_{nk}) < c_n(b_{nk}), \tag{4.46}$$

where c_n is the left-continuous slope of C_n, then C_n has a change of slope in the interval $\bar{J}_{n,k+1}$. Note that for $x \geq b_{nk}$, using the assumed smoothness of H_0, and $\inf_{[0,a]} h_0'(x) = 2\kappa > 0$,

$$V_n(x) - V_n(u_{nk}) \geq n^{-1/2}\left\{W\left(\frac{F_0(x)}{1-F_0(x)}\right) - W\left(\frac{F_0(u_{nk})}{1-F_0(u_{nk})}\right)\right\} + h_0(u_{nk})(x-u_{nk}) + \kappa(x-u_{nk})^2. \tag{4.47}$$

Now consider the event that

$$c_n(b_{nk}) \leq h_0(u_{nk}), \tag{4.48}$$

and let τ_{nk} be the first point of jump of c_n to the right of b_{nk}. Then

$$c_n(x) \leq h_0(u_{nk}), \ x < \tau_{nk},$$

and hence

$$V_n(\tau_{nk}) - V_n(x) \leq C_n(\tau_{nk}) - C_n(x) = \int_x^{\tau_{nk}} c_n(y)\,dy \leq h_0(u_{nk})(\tau_{nk} - x), \ x < \tau_{nk}.$$

Using (4.47) and stationarity of Brownian Motion, this means that the probability of (4.48) is bounded above by

$$\mathbb{P}\left\{V_n(\tau_{nk}) - V_n(u_{nk}) \leq h_0(u_{nk})(\tau_{nk} - u_{nk})\right\} \leq \mathbb{P}\left\{\exists x \geq b_{nk} : V_n(x) - V_n(u_{nk}) \leq h_0(u_{nk})(x-u_{nk})\right\}$$

$$\leq \mathbb{P}\left\{\exists x \geq b_{nk} : n^{-1/2}\left\{W\left(\frac{F_0(x)}{1-F_0(x)}\right) - W\left(\frac{F_0(u_{nk})}{1-F_0(u_{nk})}\right)\right\} \leq -\kappa(x-u_{nk})^2\right\}$$

$$= \mathbb{P}\left\{\exists x \geq b_{nk} : n^{-1/2}\left\{W\left(\frac{F_0(x)}{1-F_0(x)} - \frac{F_0(u_{nk})}{1-F_0(u_{nk})}\right)\right\} \leq -\kappa(x-u_{nk})^2\right\}. \tag{4.49}$$

We will see that this probability will become exponentially small. To this end, define the following covering of $[b_{nk}, a]$

$$K_{nkj} \stackrel{\text{def}}{=} [t_{nkj}, t_{nk,j+1}] \stackrel{\text{def}}{=} \left[b_{nk} + jn^{-1/3}, b_{nk} + (j+1)n^{-1/3}\right] = [b_{nk}, a]$$

for $0 \le j \le \lfloor n^{1/3}(a - b_{nk}) \rfloor$ (where the right end point of the last interval is taken to be a). Then the probability in (4.49) can be bounded above by

$$\sum_{j=0}^{\lfloor n^{1/3}(a-b_{nk})\rfloor} \mathbb{P}\left\{\exists x \in K_{nkj} : n^{-1/2}\left\{W\left(\frac{F_0(x)}{1-F_0(x)} - \frac{F_0(u_{nk})}{1-F_0(u_{nk})}\right)\right\} \le -\kappa(x-u_{nk})^2\right\}. \quad (4.50)$$

Denoting the probabilities in this sum by p_{nkj}, we get

$$p_{nkj} \le \mathbb{P}\left\{\sup_{x \in K_{nkj}} W\left(\frac{F_0(x)}{1-F_0(x)} - \frac{F_0(u_{nk})}{1-F_0(u_{nk})}\right) \ge \kappa\sqrt{n}(t_{nkj} - u_{nk})^2\right\}$$

$$\le \mathbb{P}\left\{\sup_{0 \le z \le F_0(t_{nk,j+1})/(1-F_0(t_{nk,j+1}))-F_0(u_{nk})/(1-F_0(u_{nk}))} W(z) \ge \kappa\sqrt{n}(t_{nkj} - u_{nk})^2\right\}.$$

Since $t_{nk,j+1} \in [b_{nk}, a]$ for all j under consideration,

$$0 \le \frac{F_0(t_{nk,j+1})}{1-F_0(t_{nk,j+1})} - \frac{F_0(u_{nk})}{1-F_0(u_{nk})} \le \frac{(F_0(t_{nk,j+1}) - F_0(u_{nk}))}{(1-F_0(a))^2} \le \lambda(t_{nk,j+1} - u_{nk})$$

for some $0 < \lambda < \infty$, we obtain, for a standard normal random variable Z

$$p_{nkj} \le \mathbb{P}\left\{\sup_{0 \le z \le \lambda(t_{nk,j+1}-u_{nk})} W(z) \ge \kappa\sqrt{n}(t_{nkj} - u_{nk})^2\right\} = \mathbb{P}\left\{|Z| \ge \frac{\kappa\sqrt{n}(t_{nkj} - u_{nk})^2}{\sqrt{\lambda(t_{nk,j+1} - u_{nk})}}\right\}$$

$$\le \mathbb{P}\left\{|Z| \ge \tilde{\kappa}\sqrt{n}(t_{nkj} - u_{nk})^{3/2}\right\} \le \frac{1}{2}\exp\left\{-\tfrac{1}{2}n\tilde{\kappa}^2(t_{nkj} - u_{nk})^3\right\}.$$

Using that $t_{nkj} - u_{nk} = b_{nk} - u_{nk} + jn^{-1/3}$, and $b_{nk} - u_{nk} \sim \frac{1}{2}n^{-1/3}\sqrt{\log n}$, we get

$$p_{nkj} \le \exp\left\{-\tfrac{1}{2}\tilde{\kappa}\left((\log n)^{3/2} + j^3\right)\right\} \Rightarrow \sum_{j=0}^{\lfloor n^{1/3}(a-b_{nk})\rfloor} p_{nkj} \le \rho \exp\left\{-\rho'(\log n)^{3/2}\right\}$$

for some $\rho, \rho' > 0$. Combining this with (4.49) and (4.50), this bounds the probability of (4.48) from above. Since a similar bound holds for the probability of the event $c_n(a_{nk}) \ge h_0(u_{nk})$, the probability that (4.46) does not hold for a specific k, is bounded by a bound of the same structure. Moreover, since this upper bound does not depend on k and $m_n \sim an^{1/3}/\log n$, the probability that there exists a $1 \le k \le m_n$ for which (4.46) does not hold satisfies the same bound (with slight change in ρ and ρ'), this proves (i). Part (ii) is an immediate consequence of (i). $\qquad \square$

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